

# Computing Black Hole entropy in Loop Quantum Gravity from a Conformal Field Theory perspective

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(Dated: March 10, 2009)

Motivated by the analogy proposed by Witten between Chern-Simons and Conformal Field Theories, we explore an alternative way of computing the entropy of a black hole starting from the isolated horizon framework in Loop Quantum Gravity. The consistency of the result opens a window for the interplay between Conformal Field Theory and the description of black holes in Loop Quantum Gravity.

PACS numbers: 04.70.Dy, 11.25.Hf, 04.60.Pp

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## I. INTRODUCTION

Ever since the pioneering work of Bekenstein [1] about the physical entropy of black holes, one of the main challenges of quantum gravity has been to describe the microscopic degrees of freedom responsible for this entropy. At the present time there are several proposals that, in spite of their totally different motivations, have reproduced the Bekenstein-Hawking law at the leading order, furthermore showing an agreement in the first order logarithmic correction. This proliferation of different alternatives raises the puzzle of understanding the underlying reason of this broad agreement. On the basis of the observation that, in addition, most of these approaches involve Conformal Field Theory (CFT) techniques at some stage, it has been suggested that conformal symmetry could play a fundamental role in this scenario (see [2] and references therein).

Loop Quantum Gravity (LQG) [3, 4] offers a detailed description of the black hole horizon quantum states [5]. In the isolated horizon framework, a black hole is introduced as an inner boundary of the spacetime manifold. Over this boundary, constraints implementing the isolated horizon properties are imposed. They reduce, already at the classical level, the  $SU(2)$  gauge symmetry of the theory to a  $U(1)$  gauge symmetry on the horizon. These  $U(1)$  degrees of freedom, that at the quantum level fluctuate independently from the ones of the bulk, are described by a Chern-Simons (CS) theory and are responsible for the horizon entropy. By counting these CS degrees of freedom, a robustly verified [6, 7, 8] linear behavior of entropy as a function of the horizon area is obtained at the leading order, showing in addition the existence of a first order logarithmic correction.

On the other hand, E. Witten proposed [9] the correspondence between the Hilbert space of generally covariant theories and the space of conformal blocks of a conformally invariant theory. This idea was applied in [10] to the computation of the entropy for a horizon described by a  $SU(2)$ -CS theory, by putting its Hilbert space in correspondence with the space of conformal blocks of a  $SU(2)$ -Wess-Zumino-Witten (WZW) model.

The purpose of the present paper is to make use of Witten's correspondence for the  $U(1)$ -CS theory describing the black hole horizon in LQG, looking for some hints on the role of CFT techniques in this framework. Taking into account the fact that this  $U(1)$  group arises as the result of a geometric symmetry breaking from the  $SU(2)$  symmetry in the bulk, one can still make use of the well established correspondence between  $SU(2)$

Chern-Simons and Wess-Zumino-Witten theories. However, in this case it will be necessary to impose restrictions on the  $SU(2)$ -WZW model in order to implement the symmetry reduction. Through this procedure we expect to eventually reproduce the counting of dimensions of the  $U(1)$ -CS Hilbert space.

## II. BLACK HOLE ENTROPY COUNTING

Let us summarize the main features and results of the black hole entropy counting in LQG in the isolated horizon framework [5]. On a space-like slice  $\Sigma$ , the geometry of the bulk is described, as usual, by a spin network. Some of the spin network edges, however, end at the horizon surface  $S$  (the intersection of  $\Sigma$  and the isolated horizon), endowing it with an area given by

$$A = 8\pi\gamma\ell_P^2 \sum_{I=1}^N \sqrt{j_I(j_I + 1)} , \quad (1)$$

where  $j_I \in \mathbb{N}/2$  label the  $SU(2)$  irreducible representations corresponding to the  $N$  edges piercing the horizon,  $\gamma$  denotes the Barbero-Immirzi parameter and  $\ell_P$  is the Planck length. These edges carry an additional label  $m_I \in \{-j_I, -j_I + 1, \dots, j_I\}$  (the corresponding spin projection) characterizing their intersection with the horizon (punctures).

On the other hand, the horizon geometry is described by a  $U(1)$  Chern-Simons theory defined over a sphere with  $N$  distinguishable topological defects (corresponding with the punctures).<sup>1</sup> The states of this theory are characterized by labels  $a_I \in \mathbb{Z}_\kappa$  ( $\kappa$  being the level of the CS theory) quantifying the angle deficits that give rise to the distributional curvature of the horizon concentrated at each puncture. The spherical topology of the horizon implies that these  $a_I$  labels must satisfy the so called projection constraint  $\sum_I a_I = 0$ . The matching of both (bulk and horizon) geometries through the boundary conditions gives rise to a relation between  $a_I$  and  $m_I$  labels, that reads

$$2m_I = -a_I \mod \kappa . \quad (2)$$

For a given value  $A$  of area, the entropy can be computed as  $S(A) = k_B \log \mathbf{n}(A)$ , being

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<sup>1</sup> The fact that punctures are distinguishable is related to the action of diffeomorphisms during the quantization procedure [3, 5], and plays a key role in the entropy counting.

$k_B$  the Boltzman constant and  $\mathbf{n}(A)$  the number of independent Chern-Simons states compatible with the above constraints, taking into account the distinguishable character of the punctures. This is to say,  $\mathbf{n}(A)$  is the number of different  $a_I$ -labeled horizon states (satisfying the projection constraint) such that, for each of them, there exists (at least) one  $(j_I, m_I)$ -labeled piercing from the bulk compatible with it and with the value  $A$  of the horizon area. The relation between  $m_I$  and  $a_I$  labels allows us then to formulate the entropy counting as a well defined combinatorial problem in terms only of the  $m_I$  labels as in [11]. Then,  $\mathbf{n}(A)$  can be rewritten as:  $\mathbf{n}(A) = 1 + \sum_{A' \leq A} d(A')$ , where  $d(A)$  is the number of all the finite, arbitrarily long, ordered sequences  $\vec{m} = (m_1, \dots, m_N)$  of non-zero half-integers, such that

$$\sum_{I=1}^N m_I = 0, \quad \sum_{I=1}^N \sqrt{|m_I|(|m_I| + 1)} = \frac{A}{8\pi\gamma\ell_P^2}. \quad (3)$$

Explicit expressions for the solution of this combinatorial problem were obtained in [8, 12]. If we define  $k_I = 2|m_I|$  and the occupancy numbers  $n_k$  as the number of punctures carrying a label value  $m$  such that  $k = 2|m|$ , then a set of numbers  $\{n_k\}$ ,  $k = 1, 2, \dots$  characterizes a  $\vec{m}$  sequence up to reorderings and sign assignments for  $m_I = \pm \frac{1}{2}k_I$ . Thus,  $d(A)$  can be expressed in terms of the set  $C$  of all the  $\{n_k\}$  sets compatible with a given area  $A$  by associating two sources of degeneracy to each of these sets  $\{n_k\}$ . The first is the number  $R(\{n_k\})$  of different ways of reordering the  $k_I$  labels in order to obtain all the corresponding ordered sequences  $\vec{k} = (k_1, \dots, k_N)$ . The second source of degeneracy is the number  $P(\{n_k\})$  of different sing assignments for the associated  $m_I$  numbers, in such a way that the projection constraint is satisfied. With this

$$d(A) = \sum_{\{n_k\} \in C} R(\{n_k\}) \times P(\{n_k\}), \quad (4)$$

where the sum is extended over all the sets  $\{n_k\}$  in  $C$ .

This set  $C$  of all  $\{n_k\}$  configurations compatible with a given area eigenvalue can be computed analytically [8] using number-theory related techniques, through an exact characterization of the horizon area spectrum of LQG. The factor  $R(\{n_k\})$  has its origin in the distinguishable character of punctures (acquired in the process of quantization of geometry) and can be obtained from basic combinatorics as  $R(\{n_k\}) = (\sum_k n_k)! / \prod_k n_k!$ , where the sum and product are extended to all values of  $k$  (note that, in practice, for a finite value  $A$  of area all the sums and products are always finite). Finally, the factor  $P(\{n_k\})$  accounts for

the dimensionality of the Hilbert space of the  $U(1)$ -CS theory once the boundary conditions have been fixed and was obtained in [8, 12] to be:

$$P(\{n_k\}) = \frac{1}{2\pi} \int_0^{2\pi} d\theta \prod_k n_k 2 \cos(k\theta) . \quad (5)$$

### III. IMPLEMENTING THE ANALOGY BETWEEN CHERN-SIMONS AND WESS-ZUMINO-WITTEN.

Let us begin by recalling the classical scenario and how the symmetry reduction takes place at this level. The geometry of the bulk is described by a  $SU(2)$  connection, whose restriction to the horizon  $S$  gives rise to a  $SU(2)$  connection over this surface. As a consequence of imposing the isolated horizon boundary conditions this connection is reduced to a  $U(1)$  connection. In [5] this reduction is carried out, at the classical level, just by fixing a unit vector  $\vec{r}$  at each point of the horizon. By defining a smooth function  $r : S \rightarrow su(2)$  a  $U(1)$  sub-bundle is picked out from the  $SU(2)$  bundle. This kind of reduction can be described in more general terms as follows (see, for instance, [13]). Let  $P(SU(2), S)$  be a  $SU(2)$  principal bundle over the horizon, and  $\omega$  the corresponding connection over it. A homomorphism  $\lambda$  between the closed subgroup  $U(1) \subset SU(2)$  and  $SU(2)$  induces a bundle reduction from  $P(SU(2), S)$  to  $Q(U(1), S)$ , being  $Q$  the resulting  $U(1)$  principal bundle with reduced  $U(1)$  connection  $\omega'$ . This  $\omega'$  is obtained, in this case, from the restriction of  $\omega$  to  $U(1)$ .

All the conjugacy classes of homomorphisms  $\lambda : U(1) \rightarrow SU(2)$  are represented in the set  $Hom(U(1), T(SU(2)))$ , where  $T(SU(2)) = \{diag(z, z^{-1}) | z = e^{i\theta} \in U(1)\}$  is the maximal torus of  $SU(2)$ . The homomorphisms in  $Hom(U(1), T(SU(2)))$  can be characterized by

$$\lambda_p : z \mapsto diag(z^p, z^{-p}) , \quad (6)$$

for any  $p \in \mathbb{Z}$ . However the generator of the Weyl group of  $SU(2)$  acts on  $T(SU(2))$  by  $diag(z, z^{-1}) \mapsto diag(z^{-1}, z)$ . If we divide out by the action of the Weyl group we are just left with those maps  $\lambda_p$  with  $p$  a non-negative integer,  $p \in \mathbb{N}_0$ , as representatives of all conjugacy classes. These  $\lambda_p$  characterize then all the possible ways to carry out the symmetry breaking from the  $SU(2)$  to the  $U(1)$  connection that will be quantized later.

However, one can follow the alternative approach of first quantizing the  $SU(2)$  connection

on  $S$  and imposing the boundary conditions later on, at the quantum level. This would give rise to a  $SU(2)$ -CS theory on the horizon to which the boundary conditions have to be imposed. The correspondence with conformal field theories can be used at this point to compute the dimension of the Hilbert space of the  $SU(2)$ -CS as the number of conformal blocks of the  $SU(2)$ -WZW model, as it was done in [10]. It is necessary to require, then, additional restrictions to the  $SU(2)$ -WZW model that account for the symmetry breaking, and consider only the degrees of freedom corresponding to a  $U(1)$  subgroup.

Let us briefly review the computation in the  $SU(2)$  case, to later introduce the symmetry reduction. The number of conformal blocks of the  $SU(2)$ -WZW model<sup>2</sup>, given a set of representations  $\mathcal{P} = \{j_1, j_2, \dots, j_N\}$ , can be computed in terms of the so-called fusion numbers  $N_{il}^r$  [14] as

$$N^{\mathcal{P}} = \sum_{r_i} N_{j_1 j_2}^{r_1} N_{r_1 j_3}^{r_2} \dots N_{r_{N-2} j_{N-1}}^{j_N} . \quad (7)$$

These  $N_{il}^r$  are the number of independent couplings between three primary fields, *i.e.* the multiplicity of the  $r$ -irreducible representation in the decomposition of the tensor product of the  $i$  and  $l$  representations  $[j_i] \otimes [j_l] = \bigoplus_r N_{il}^r [j_r]$ . This expression is known as a fusion rule.  $N^{\mathcal{P}}$  is then the multiplicity of the  $SU(2)$  gauge invariant representation ( $j = 0$ ) in the decomposition of the tensor product  $\bigotimes_{i=1}^N [j_i]$  of the representations in  $\mathcal{P}$ . The usual way of computing  $N^{\mathcal{P}}$  is using the Verlinde formula [14] to obtain the fusion numbers. But alternatively one can make use of the fact that the characters of the  $SU(2)$  irreducible representations,  $\chi_i = \sin[(j_i + 1)\theta]/\sin\theta$ , satisfy the fusion rules  $\chi_i \chi_j = \sum_r N_{ij}^r \chi_r$ . Taking into account that the characters form an orthonormal set with respect to the  $SU(2)$  scalar product,  $\langle \chi_i | \chi_j \rangle = \delta_{ij}$ , one can obtain the number of conformal blocks just by projecting the product of characters over the gauge invariant representation

$$N^{\mathcal{P}} = \langle \chi_{j_1} \dots \chi_{j_N} | \chi_0 \rangle = \int_0^{2\pi} \frac{d\theta}{\pi} \sin^2 \theta \prod_{I=1}^N \frac{\sin[(j_I + 1)\theta]}{\sin \theta} . \quad (8)$$

This expression is equivalent to the one obtained in [10] using the Verlinde formula; it gives rise to the same result for every set of punctures  $\mathcal{P}$ .

To implement, now, the symmetry breaking we have to restrict the representations in  $\mathcal{P}$  to a set of  $U(1)$  representations. This corresponds in the case of Chern-Simons theory to

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<sup>2</sup> Notice that, though we are omitting the  $\kappa$  subindex, the group of the WZW theory is in fact the quantum group  $SU(2)_{\kappa}$ . The  $\kappa$  dependence is implicit in the allowed sets of representations  $\mathcal{P}$ .

performing a symmetry reduction locally at each puncture. It is known that each  $SU(2)$  irreducible representation  $j$  contains the direct sum of  $2j + 1$   $U(1)$  representations  $e^{ij\theta} \oplus e^{i(j-1)\theta} \oplus \dots \oplus e^{-ij\theta}$ . One can make an explicit symmetry reduction by just choosing one of the possible restrictions of  $SU(2)$  to  $U(1)$  which, as we saw above, are given by the homomorphisms  $\lambda_p$ . This corresponds here to pick out a  $U(1)$  representation of the form  $e^{ip\theta} \oplus e^{-ip\theta}$  with some  $p \leq j$ . The fact that we will be using these reducible representations, consisting of  $SU(2)$  elements as  $U(1)$  representatives, can be seen as a reminiscence from the fact that the  $U(1)$  freedom has its origin in the reduction from  $SU(2)$ .

Having implemented the symmetry reduction, let us compute the number of independent couplings in this  $U(1)$ -reduced case. Of course, we are considering now  $U(1)$  invariant couplings, so we have to compute the multiplicity of the  $m = 0$  irreducible  $U(1)$  representation in the direct sum decomposition of the tensor product of the representations involved. As in the previous case, this can be done by using the characters of the representations and the fusion rules they satisfy. These characters can be expressed as  $\tilde{\eta}_{p_I} = e^{ip_I\theta} + e^{-ip_I\theta} = 2 \cos p_I\theta$ . Again, we can make use of the fact that the characters  $\eta_i$  of the  $U(1)$  irreducible representations are orthonormal with respect to the standard scalar product in the circle. Then, the number we are looking for is given by

$$N_{U(1)}^{\mathcal{P}} = \langle \tilde{\eta}_{p_1} \dots \tilde{\eta}_{p_N} | \eta_0 \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\theta \prod_I^N 2 \cos p_I\theta, \quad (9)$$

where  $\eta_0 = 1$  is the character of the  $U(1)$  gauge invariant irreducible representation. We can see that this result is exactly the same as the one obtained for  $P(\{n_k\})$  in Eq. (5), coming from the  $U(1)$ -CS theory, just by identifying the  $p_I$  with  $k_I$  labels.

#### IV. REMARKS AND CONCLUSIONS

Let us put this result in context with the entropy counting. As explained above, in computing the entropy of a black hole within LQG, there are several contributions involved. Some of them are related with the LQG framework, like the computation of  $C$  by characterizing the black hole area spectrum, or  $R(\{n_k\})$  due to the distinguishability of the punctures originated in the quantization process. The term  $P(\{n_k\})$ , however, is related with the CS theory on the horizon. Once one has introduced all the conditions imposed by the LQG

framework, what is left is the counting of states of a CS theory subject to some external inputs. If there is any connection between this CS theory and a Conformal Field Theory, one should expect this CFT to reproduce precisely this term  $P(\{n_k\})$ , subject to the same external inputs. This is exactly what we observe here by identifying the  $p_I$  and  $k_I$  labels. We are, thus, proposing a precise implementation of Witten's analogy through this symmetry reduced counting that yields the expected result.

From the physical point of view, the main change we are introducing, besides using the CS-CFT analogy, is to impose the isolated horizon boundary conditions at the quantum level, instead of doing it prior to the quantization process. This is a preliminary step in the direction of introducing a quantum definition of isolated horizons.

It is very interesting to observe that, as shown in [15], the contribution to entropy of  $P(\{n_k\})$  has a linear growth with area, including a logarithmic correction. The  $R(\{n_k\})$  term, on the other hand, introduces very particular quantum effects, that are specific from LQG. In particular, the stair-like behavior appearing at the Planck scale in the entropy-area relation [7, 8], has its origin in this factor. Thus, the picture obtained here seems to be compatible with the proposed role of conformal symmetry on the first order behavior of black hole entropy.

### Acknowledgment

We thank V. Aldaya, A. Ashtekar, J. A. de Azcarraga, R. Coquereaux and L. Freidel for interesting discussions and suggestions. We specially thank F. Barbero and E. Villasenor for many discussions, strong encouragement, and a careful reading of the manuscript. I. A. thanks R. Wald for his kind hospitality at the University of Chicago where part of this work was done. J. D. thanks A. Ashtekar for his kind hospitality at the IGC at Penn State during the realization of part of this work.

This work was in part supported by Spanish grants FIS2008-06078-C03-02, FIS2008-01980 and ESP2005-07714-C03-0, the NSF grant PHY0854743 and the Eberly research funds of Penn State, and the NSF grant PHY04-56619 to the University of Chicago. I. A. and J. D. acknowledge financial support provided by the Spanish Ministry of Science and Education



under the FPU program.

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